

PD control of a rotating smart beam with an elastic root

Shueei-Muh Lin

Mechanical Engineering Department, Kun Shan University, Tainan, 710-03, Taiwan, Republic of China

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Abstract

The paper is aimed at the proportional and derivative controls of vibration of a rotating beam by using a pair of piezoelectric sensor and actuator layers. Using Hamilton's principle derives the governing differential equations and the boundary conditions for the coupled axial-bending vibration of a piezoelectric rotating beam with elastically restrained root. The analytical method given by Lin et al. is used to determine the transient response of a piezoelectric rotating beam. The influences of the proportional and derivative control gain factors on the performance of the first two modes of an elastically constrained beam are investigated. It is found that considering the proportional control law only is not helpful to the active damping of a rotating beam. Moreover, considering the proportional and derivative controls suitably and simultaneously enhances the active damping of a rotating beam with an elastic root.

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1. Introduction

Rotating beams, which have importance in many practical applications such as turbine blades, helicopter rotor blades, airplane propellers, and robot manipulators, have been investigated for a long time. An interesting review of the subject can be found in the papers by Leissa [1], Ramamurti and Balasubramanian [2], Rao [3], and Lin [4]. Much attention has been focused on the undamped vibration problems. Lin et al. [5] and Lin and Lee [6] studied the passive damping of a rotating beam. Lin et al. [7] studied the active damping of a rotating cantilever beam by using the derivative control law. So far, little research has been done on the active-damping problem of a rotating smart beam because of its complexity.

Turcotte et al. [8] studied the vibration of a mistuned bladed-disk assembly using nonrotational structurally damped beams. The structural damping was introduced through a complex bending rigidity. Patel and Ganapathi [9] studied the free torsional vibration of nonrotating damped sandwich beams. Friswell and Lees [10] studied the free vibration of simply supported nonrotating damped beams. Lin et al. [5,6] investigated the vibration and instability of a rotating structurally and viscously damped beam with an elastically restrained root and root damping. The complex frequency relations among different systems were revealed. The instability of divergence, oscillating and non-oscillating motions were predicted exactly via the relations. The above literature investigated the passive damped vibration problems. Piezoelectric materials have been applied to the active control of structural vibrations and noises. Owing to the complexity of analytical methods the

E-mail address: smlin@mail.ksu.edu.tw

Nomenclature			
a	$a_a + a_h + a_s$	u, v, w displacements in the $x, y,$ and z directions, respectively	
a_i	dimensionless rotary inertia per unit length, $\rho_i I_i / (\rho_h A_h L^2)$	w_b width of beam	
A_i	cross-sectional area of the beam, $w_b \int_{z_i, \text{bottom}}^{z_i, \text{upper}} dz$	W dimensionless lateral displacements in the z direction, w/L	
b	$b_a + b_h + b_s$	x, y, z principal frame coordinates of blade	
b_i	dimensionless bending rigidity, $E_i I_i / (E_h I_h)$	z_c the neutral axis, $\sum_{i=a,h,s} E_i h_i [(\sum_{j=a}^i h_j) - \frac{1}{2} h_i] / \sum_{i=a,h,s} E_i h_i$	
B_i	$w_b \int_{z_i, \text{bottom}}^{z_i, \text{upper}} z dz$	z_s $h_1 + h_2 + (h_3/2) - z_c$	
c	elastic stiffness constant	α dimensionless rotational speed, $\Omega L^2 / \sqrt{\rho_h A_h / (E_h I_h)}$	
d_{qd}, d_{qp}	dimensionless derivative piezoelectric parameter, $[-e_{31,a} e_{31,s} B_a z_s h_s / (\mu_{33}^e L^2 h_a \sqrt{\rho_h A_h E_h I_h})] k_i, i = d, p$	β_T, β_θ dimensionless translational and rotational spring constants, $\beta_T = K_T L^3 / E_2 I_2, \beta_\theta = K_\theta L / E_2 I_2$	
D	electric displacement $[\mu_{33}^e A_a z_s^2 (h_s e_{31,s} / (\mu_{33}^e L h_a))^2 / (\rho_h A_h L^2)] k_i^2, i = d, p$	γ spring constant, $\gamma_{11} = \beta_T / (1 + \beta_T), \gamma_{12} = 1 / (1 + \beta_T), \gamma_{21} = \beta_\theta / (1 + \beta_\theta), \gamma_{22} = 1 / (1 + \beta_\theta)$	
\bar{e}_i	dimensionless bending rigidity, $a E_i I_i / (E_h I_h)$	ε strain	
\bar{e}	$\bar{e}_a + \bar{e}_h + \bar{e}_s$	ζ decay rate	
e	piezoelectric constant	θ setting angle	
E	Young's modulus of beam	μ permittivity	
E_p	electric field intensity	ξ dimensionless distance to the root of the beam, x/L	
EA	$E_a A_a + E_h A_h + E_s A_s$	ρ mass density per unit volume of beam	
EB	$E_a B_a + E_s B_s$	ρ_A $\rho_a A_a + \rho_h A_h + \rho_s A_s$	
EI	$E_a I_a + E_h I_h + E_s I_s$	ρ_B $\rho_a B_a + \rho_h B_h + \rho_s B_s$	
H_i	$\mu_{33}^e [h_s e_{31,s} k_i / (\mu_{33}^e L h_a)]^2, i = d, p$	ρ_I $\rho_a I_a + \rho_h I_h + \rho_s I_s$	
I_i	area moment inertia of the beam, $w_b \int_{z_i, \text{bottom}}^{z_i, \text{upper}} z^2 dz$	σ stress	
j	imaginary unit	τ dimensionless time, $t \sqrt{E_h I_h / (\rho_h A_h L^4)}$	
K_T, K_θ	translational and rotational spring constants, respectively	ϖ frequency	
L	length of the blade	Ω rotational speed	
m	$m_s + m_h + m_a$	ω dimensionless frequency, $\varpi \sqrt{\rho_h A_h L^4 / (E_h I_h)}$	
m_i	dimensionless mass per unit length, $\rho_i A_i / (\rho_h A_h)$	<i>Subscripts</i>	
n	dimensionless centrifugal force, $\alpha^2 \int_{\xi}^1 m (r + \chi) d\chi$	a actuator	
N	centrifugal force	h host beam	
Q_d, Q_p	$[-e_{31,a} e_{31,s} h_s / (\mu_{33}^e L^2 h_a)] k_i, i = d, p$	s sensor	
r	dimensionless radius of root, R/L	1, 2, 3 in the x -, y - and z -direction, respectively	
R	radius of root	<i>Superscripts</i>	
t	time variable	*	independent of '**'

approximated finite element method has been investigated by many researchers [11,12]. So far, little research has been done on the active-damping problem of a piezoelectric rotating beam because of its complexity.

Lin et al. [7] investigated the active damping of the first mode of a cantilever beam under a derivative control law. In this paper, it uses Hamilton's principle to derive the governing differential equations and the boundary conditions for the coupled axial-bending vibration of a piezoelectric rotating beam with an elastic

root. The proportional and derivative control laws are simultaneously used to control the performance of a rotating beam with an elastic root. The analytical method given by Lin et al. [7] is used to determine the transient response of the system. Finally, the influence of the proportional and derivative gain factors, the rotational and translational spring constants and the rotating speed on the natural frequencies and the decay rate are investigated.

2. Governing equations and boundary and initial conditions

Consider the transient response of a piezoelectric rotating beam mounted with setting angle θ on a hub with radius R , rotating with constant angular velocity Ω . The upper and bottom surfaces of beam are bonded sensor and actuator layers, as shown in Fig. 1. A set of differential equations of coupled axial-bending motion for the rotating beam are derived based on the following assumptions [7]:

- (1) The beam is assumed to be narrow in both the y - and z -direction, and not loaded in these directions, then $\sigma_2 = \sigma_3 = 0$.
- (2) The shear deformation is negligible.
- (3) The rotary inertia is considered.
- (4) The transverse displacement is same for all three layers.
- (5) Linear theory of piezoelectricity is applicable.
- (6) The electric field will be applied to the piezoelectric actuator on the z -direction (perpendicular to the planes of piezoelectric film). Therefore $E_{p1} = E_{p2} = 0$.

The displacement fields of the beam are

$$u = u_0(x, t) - z \frac{\partial w(x, t)}{\partial x}, \quad v = 0, \quad w = w(x, t). \quad (1)$$

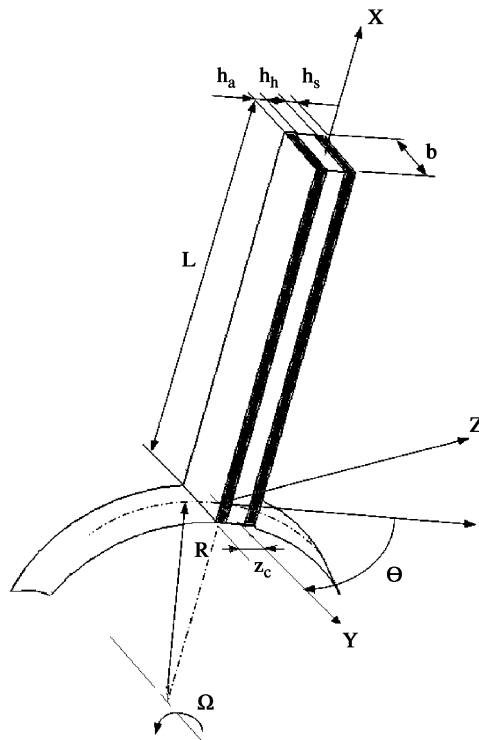


Fig. 1. Geometry and coordinate system of a rotating beam.

Based on the effect of centrifugal force, the nonlinear term of strain of the host beam is considered:

$$\varepsilon_1 = \frac{\partial u_0}{\partial x} - z \frac{\partial^2 w}{\partial x^2} + \frac{1}{2} \left(\frac{\partial w}{\partial x} \right)^2, \quad \varepsilon_2 = 0, \quad \varepsilon_3 = 0. \quad (2)$$

The kinetic energy T of the beam is

$$T = \frac{1}{2} \int_0^L \int_A \vec{V} \cdot \vec{V} \rho \, dA \, dx, \quad (3)$$

where the velocity vector of a point (x, y, z) in the beam is

$$\begin{aligned} \vec{V} = & \left[\frac{\partial u}{\partial t} + \Omega \sin \theta (z + w) + y \Omega \cos \theta \right] i \\ & + [(x + R + u) \Omega \cos \theta] j + \left[\frac{\partial w}{\partial t} - (x + R + u) \Omega \sin \theta \right] k. \end{aligned} \quad (4)$$

The extended potential energy including the electric contribution is

$$U = \frac{1}{2} \int_v \sigma_1 \varepsilon_1 \, dv - \frac{1}{2} \int_v E_{p3} D_3 \, dv + \frac{1}{2} K_T w^2(0, t) + \frac{1}{2} K_\theta \left[\frac{\partial w(0, t)}{\partial x} \right]^2. \quad (5)$$

The constitutive equation of the piezoelectric material is

$$\begin{aligned} \sigma_1 &= c_{11}^E \varepsilon_1 - e_{31} E_{p3}, \\ D_3 &= e_{31} \varepsilon_1 + \mu_{33}^E E_{p3}. \end{aligned} \quad (6)$$

The piezoelectric layer is used to sense the vibration of the rotating beam. The charge accumulated on the layer due to the direct piezoelectric effect is evaluated by

$$q = w_b \int e_{31} \varepsilon_1 \, dx. \quad (7)$$

Considering the sensor to be a parallel capacitor, the voltage of the sensor is

$$V_s = \frac{h_s}{\mu_{33}^E L} \int e_{31} \varepsilon_1 \, dx. \quad (8)$$

In closed-loop control, the control voltage on the piezoelectric actuator is designed by the following proportional and derivative control laws [13]

$$V_a = -k_p V_s - k_d \frac{\partial V_s}{\partial t}, \quad (9)$$

where k_d and k_p are the derivative and proportional gain factors.

Application of Hamilton's principle yields the following governing differential equations:

$$\begin{aligned} & -\rho A \frac{\partial^2 u_0}{\partial t^2} + \rho B \frac{\partial}{\partial x} \left(\frac{\partial^2 w}{\partial t^2} \right) - 2\rho A \Omega \sin \theta \frac{\partial w}{\partial t} - \rho B \Omega^2 \frac{\partial w}{\partial x} \\ & + \rho A \Omega^2 (x + u_0 + R) + \frac{\partial N}{\partial x} + EA \frac{\partial^2 u_0}{\partial x^2} - EB \frac{\partial^3 w}{\partial x^3} - Q_p (B_a - A_a z_s) \frac{\partial^2 w}{\partial x^2} \\ & - 2Q_d A_a \frac{\partial^2 u_0}{\partial x \partial t} + Q_d (B_a + A_a z_s) \frac{\partial^3 w}{\partial t \partial x^2} + H_p \left(A_a u_0 - A_a z_s \frac{\partial w}{\partial x} \right) \\ & - H_d \left(A_a \frac{\partial^2 u_0}{\partial t^2} - A_a z_s \frac{\partial^3 w}{\partial t^2 \partial x} \right) = 0, \end{aligned} \quad (10)$$

$$\begin{aligned}
& -\rho B \frac{\partial^2}{\partial t^2} \left(\frac{\partial u_0}{\partial x} \right) + \rho I \frac{\partial^2}{\partial t^2} \left(\frac{\partial^2 w}{\partial x^2} \right) + \rho B \Omega^2 \frac{\partial u_0}{\partial x} - \rho I \Omega^2 \frac{\partial^2 w}{\partial x^2} + 2\rho A \Omega \sin \theta \frac{\partial u_0}{\partial t} \\
& - 4\rho B \Omega \sin \theta \frac{\partial^2 w}{\partial t \partial x} - \rho A \frac{\partial^2 w}{\partial t^2} + \rho B \Omega^2 \sin^2 \theta + \rho A w \Omega^2 \sin^2 \theta \\
& + \frac{\partial}{\partial x} \left(N \frac{\partial w}{\partial x} \right) + EB \frac{\partial^3 u_0}{\partial x^3} - EI \frac{\partial^4 w}{\partial x^4} - Q_p (B_a - A_a z_s) \frac{\partial^2 u_0}{\partial x^2} \\
& - Q_d \left[(B_a + A_a z_s) \frac{\partial^3 u_0}{\partial x^2 \partial t} - 2B_a z_s \frac{\partial^4 w}{\partial t \partial x^3} \right] + H_p \left(A_a z_s \frac{\partial u_0}{\partial x} - A_a z_s^2 \frac{\partial^2 w}{\partial x^2} \right) \\
& - H_d \left(A_a z_s \frac{\partial^3 u_0}{\partial x \partial t^2} - A_a z_s^2 \frac{\partial^4 w}{\partial t^2 \partial x^2} \right) = 0
\end{aligned} \tag{11}$$

and the associated boundary conditions.

At $x = 0$,

$$u_0 = 0, \tag{12}$$

$$\begin{aligned}
& EB \frac{\partial u_0}{\partial x} - EI \frac{\partial^2 w}{\partial x^2} - Q_p \left(B_a u_0 - B_a z_s \frac{\partial w}{\partial x} \right) \\
& - Q_d \left(B_a \frac{\partial u_0}{\partial t} - B_a z_s \frac{\partial^2 w}{\partial t \partial x} \right) + K_\theta \frac{\partial w}{\partial x} = 0,
\end{aligned} \tag{13}$$

$$\begin{aligned}
& \rho B \frac{\partial^2 u_0}{\partial t^2} - \rho I \frac{\partial^2}{\partial t^2} \left(\frac{\partial w}{\partial x} \right) + \rho I \Omega^2 \frac{\partial w}{\partial x} - \rho B \Omega^2 (x + u_0 + R) \\
& + 2\rho B \Omega \sin \theta \frac{\partial w}{\partial t} - N \frac{\partial w}{\partial x} - EB \frac{\partial^2 u_0}{\partial x^2} + EI \frac{\partial^3 w}{\partial x^3} \\
& + Q_p (B_a - A_a z_s) \frac{\partial u_0}{\partial x} + Q_d \left[(B_a + A_a z_s) \frac{\partial^2 u_0}{\partial x \partial t} - 2B_a z_s \frac{\partial^3 w}{\partial t \partial x^2} \right] \\
& - H_p \left(A_a z_s u_0 - A_a z_s^2 \frac{\partial w}{\partial x} \right) + H_d \left(A_a z_s \frac{\partial^2 u_0}{\partial t^2} - A_a z_s^2 \frac{\partial^3 w}{\partial t^2 \partial x} \right) + K_T w = 0.
\end{aligned} \tag{14}$$

At $x = L$,

$$N + EA \frac{\partial u_0}{\partial x} - EB \frac{\partial^2 w}{\partial x^2} - Q_p \left(A_a u_0 - A_a z_s \frac{\partial w}{\partial x} \right) - Q_d \left(A_a \frac{\partial u_0}{\partial t} - A_a z_s \frac{\partial^2 w}{\partial t \partial x} \right) = 0, \tag{15}$$

$$EB \frac{\partial u_0}{\partial x} - EI \frac{\partial^2 w}{\partial x^2} - Q_p \left(B_a u_0 - B_a z_s \frac{\partial w}{\partial x} \right) - Q_d \left(B_a \frac{\partial u_0}{\partial t} - B_a z_s \frac{\partial^2 w}{\partial t \partial x} \right) = 0, \tag{16}$$

$$\begin{aligned}
& \rho B \frac{\partial^2 u_0}{\partial t^2} - \rho I \frac{\partial^2}{\partial t^2} \left(\frac{\partial w}{\partial x} \right) + \rho I \Omega^2 \frac{\partial w}{\partial x} - \rho B \Omega^2 (x + u_0 + R) \\
& + 2\rho B \Omega \sin \theta \frac{\partial w}{\partial t} - N \frac{\partial w}{\partial x} - EB \frac{\partial^2 u_0}{\partial x^2} + EI \frac{\partial^3 w}{\partial x^3} \\
& + Q_p (B_a - A_a z_s) \frac{\partial u_0}{\partial x} + Q_d \left[(B_a + A_a z_s) \frac{\partial^2 u_0}{\partial x \partial t} - 2B_a z_s \frac{\partial^3 w}{\partial t \partial x^2} \right] \\
& - H_p \left(A_a z_s u_0 - A_a z_s^2 \frac{\partial w}{\partial x} \right) + H_d \left(A_a z_s \frac{\partial^2 u_0}{\partial t^2} - A_a z_s^2 \frac{\partial^3 w}{\partial t^2 \partial x} \right) = 0.
\end{aligned} \tag{17}$$

It should be noted that there exist the terms in Eqs. (11), (14), (17), $\{\rho B \Omega^2 \sin^2 \theta, \rho B \Omega^2 (x + u_0 + R)\}$, independent of time. The first term represents a small transverse static centrifugal force due to the coupled effect of the rotational speed and the small difference between the geometry center and the neutral one at

which the bending stress is zero. The last term represents a static axial centrifugal force. Because these will not affect the dynamic behavior, the static term is negligible in studying the dynamic behavior of beam.

Because the bending motion dominates in the vibration of a beam and the natural frequency of the longitudinal motion is greatly higher than that of the bending motion, the effect of longitudinal displacement is negligible. Moreover, the lateral vibration of a blade subjected to low rotational speed is dominant and the effect of the Coriolis force may be neglected [4]. In this paper, assume that the beam is in extensional and the effect of the Coriolis force, $2\rho\Omega \sin \theta(\partial w/\partial t)$ of Eqs. (10), (14), (17), is neglected. Moreover, because the sensor and actuator layers are considered to be thin, the axial force in Eq. (10) is dominated by the centrifugal force, $\rho A\Omega^2(x+R)$. As a result, the centrifugal force N can be expressed as

$$N(x) = \Omega^2 \int_x^L \rho(x)A(x)(R+x) dx. \quad (18)$$

The governing equation in terms of the transverse displacement w , becomes

$$\begin{aligned} \rho I \frac{\partial^2}{\partial t^2} \left(\frac{\partial^2 w}{\partial x^2} \right) - \rho I \Omega^2 \frac{\partial^2 w}{\partial x^2} - \rho A \frac{\partial^2 w}{\partial t^2} + \rho A \Omega^2 \sin^2 \theta w + \frac{\partial}{\partial x} \left(N \frac{\partial w}{\partial x} \right) \\ - EI \frac{\partial^4 w}{\partial x^4} + 2Q_d B_a z_s \frac{\partial^4 w}{\partial t \partial x^3} - H_p A_a z_s^2 \frac{\partial^2 w}{\partial x^2} + H_d A_a z_s^2 \frac{\partial^4 w}{\partial t^2 \partial x^2} = 0. \end{aligned} \quad (19)$$

The boundary conditions at $x = 0$ are

$$\begin{aligned} -EI \frac{\partial^2 w}{\partial x^2} + Q_p B_a z_s \frac{\partial w}{\partial x} + Q_d B_a z_s \frac{\partial^2 w}{\partial t \partial x} + K_\theta \frac{\partial w}{\partial x} = 0, \quad (20) \\ -\rho I \frac{\partial^2}{\partial t^2} \left(\frac{\partial w}{\partial x} \right) + \rho I \Omega^2 \frac{\partial w}{\partial x} - N \frac{\partial w}{\partial x} + EI \frac{\partial^3 w}{\partial x^3} \\ - 2Q_d B_a z_s \frac{\partial^3 w}{\partial t \partial x^2} + H_p A_a z_s^2 \frac{\partial w}{\partial x} - H_d A_a z_s^2 \frac{\partial^3 w}{\partial t^2 \partial x} + K_T w = 0 \end{aligned} \quad (21)$$

and at $x = L$,

$$\begin{aligned} -EI \frac{\partial^2 w}{\partial x^2} + Q_p B_a z_s \frac{\partial w}{\partial x} + Q_d B_a z_s \frac{\partial^2 w}{\partial t \partial x} = 0, \quad (22) \\ -\rho I \frac{\partial^2}{\partial t^2} \left(\frac{\partial w}{\partial x} \right) + \rho I \Omega^2 \frac{\partial w}{\partial x} - N \frac{\partial w}{\partial x} + EI \frac{\partial^3 w}{\partial x^3} \\ - 2Q_d B_a z_s \frac{\partial^3 w}{\partial t \partial x^2} + H_p A_a z_s^2 \frac{\partial w}{\partial x} - H_d A_a z_s^2 \frac{\partial^3 w}{\partial t^2 \partial x} = 0. \end{aligned} \quad (23)$$

It should be noted that when the sensor and the actuator are neglected, the differential equations are the same as those given by Lin [14].

In terms of dimensionless quantities listed the nomenclature, the governing differential equation (19) and the boundary conditions (20)–(23) of the system are non-dimensionalized as follows:

$$\begin{aligned} a \frac{\partial^4 W}{\partial \xi^2 \partial \tau^2} - a x^2 \frac{\partial^2 W}{\partial \xi^2} - m \frac{\partial^2 W}{\partial \tau^2} + m W \alpha^2 \sin^2 \theta + \frac{\partial}{\partial \xi} \left(n \frac{\partial W}{\partial \xi} \right) \\ - \bar{e} \frac{\partial^4 W}{\partial \xi^4} - d_{hp} \frac{\partial^2 W}{\partial \xi^2} + 2d_{qd} \frac{\partial^4 W}{\partial \xi^3 \partial \tau} + d_{hd} \frac{\partial^4 W}{\partial \xi^2 \partial \tau^2} = 0. \end{aligned} \quad (24)$$

At $\xi = 0$,

$$\begin{aligned} & -\gamma_{12}a \frac{\partial^3 W}{\partial \xi \partial \tau^2} + \gamma_{12}a\alpha^2 \frac{\partial W}{\partial \xi} - \gamma_{12}n \frac{\partial W}{\partial \xi} + \gamma_{12}\bar{e} \frac{\partial^3 W}{\partial \xi^3} \\ & - 2\gamma_{12}d_{qd} \frac{\partial^3 W}{\partial \xi^2 \partial \tau} + \gamma_{12}d_{hp} \frac{\partial w}{\partial \xi} - \gamma_{12}d_{hd} \frac{\partial^3 W}{\partial \xi \partial \tau^2} + \gamma_{11}W = 0, \end{aligned} \quad (25)$$

$$-\gamma_{22}\bar{e} \frac{\partial^2 W}{\partial \xi^2} + \gamma_{22}d_{qp} \frac{\partial w}{\partial \xi} + \gamma_{22}d_{qd} \frac{\partial^2 W}{\partial \xi \partial \tau} + \gamma_{21} \frac{\partial W}{\partial \xi} = 0 \quad (26)$$

and at $\xi = 1$,

$$-\bar{e} \frac{\partial^2 W}{\partial \xi^2} + d_{qp} \frac{\partial w}{\partial \xi} + d_{qd} \frac{\partial^2 W}{\partial \xi \partial \tau} = 0, \quad (27)$$

$$-a \frac{\partial^3 W}{\partial \xi \partial \tau^2} + \alpha^2 \frac{\partial W}{\partial \xi} - n \frac{\partial W}{\partial \xi} + \bar{e} \frac{\partial^3 W}{\partial \xi^3} - 2d_{qd} \frac{\partial^3 W}{\partial \xi^2 \partial \tau} + d_{hp} \frac{\partial w}{\partial \xi} - d_{hd} \frac{\partial^3 W}{\partial \xi \partial \tau^2} = 0. \quad (28)$$

The dimensionless initial conditions of the motion at the tip are

$$W(1, 0) = w_0 \quad \text{and} \quad \frac{\partial W(1, 0)}{\partial \tau} = \dot{w}_0. \quad (29)$$

3. Solution method

3.1. Characteristic governing equations and boundary conditions

Assume that the solution to Eqs. (24)–(29) is

$$W(\xi, \tau) = \tilde{W}(\xi)e^{\lambda\tau}, \quad (30a)$$

where \tilde{W} represents the complex mode function and λ is the complex frequency. They can be expressed as

$$\tilde{W}(\xi) = \bar{W}_R(\xi) + j\bar{W}_I(\xi), \quad \lambda = -\zeta + j\omega, \quad (30b)$$

where ζ is the decay rate. The imaginary term ω is the damped frequency.

Letting $\text{Re } W(1, 0) = w_0$, and $\text{Re } \dot{W}(1, 0) = \dot{w}_0$, one obtains

$$W_R(1) = w_0 \quad \text{and} \quad \bar{W}_I(1) = \frac{-1}{\omega}[\zeta w_0 + \dot{w}_0]. \quad (31)$$

Substituting Eqs. (30) into the governing equation (24) and the boundary conditions (25)–(28) and taking the real and imaginary parts apart, the coupled real differential equations can be obtained:

$$\begin{aligned} & -\bar{e} \frac{d^4 \bar{W}_R}{d\xi^4} - 2d_{qd} \left(\zeta \frac{d^3 \bar{W}_R}{d\xi^3} + \omega \frac{d^3 \bar{W}_I}{d\xi^3} \right) + a \left[(\zeta^2 - \omega^2 - \alpha^2) \frac{d^2 \bar{W}_R}{d\xi^2} + 2\zeta\omega \frac{d^2 \bar{W}_I}{d\xi^2} \right] \\ & + n \frac{d^2 \bar{W}_R}{d\xi^2} - d_{hp} \frac{d^2 \bar{W}_R}{d\xi^2} + d_{hd} \left[(\zeta^2 - \omega^2) \frac{d^2 \bar{W}_R}{d\xi^2} + 2\zeta\omega \frac{d^2 \bar{W}_I}{d\xi^2} \right] \\ & + \frac{dn}{d\xi} \frac{d\bar{W}_R}{d\xi} - m[(\zeta^2 - \omega^2 - \alpha^2 \sin^2 \theta) \bar{W}_R + 2\zeta\omega \bar{W}_I] = 0, \end{aligned} \quad (32a)$$

$$\begin{aligned}
& -\bar{e} \frac{d^4 \bar{W}_I}{d\xi^4} - 2d_{qd} \left(\zeta \frac{d^3 \bar{W}_I}{d\xi^3} - \omega \frac{d^3 \bar{W}_R}{d\xi^3} \right) + a \left[(\zeta^2 - \omega^2 - \alpha^2) \frac{d^2 \bar{W}_I}{d\xi^2} - 2\zeta\omega \frac{d^2 \bar{W}_R}{d\xi^2} \right] \\
& + n \frac{d^2 \bar{W}_I}{d\xi^2} - d_{hp} \frac{d^2 \bar{W}_I}{d\xi^2} + d_{hd} \left[(\zeta^2 - \omega^2) \frac{d^2 \bar{W}_I}{d\xi^2} - 2\zeta\omega \frac{d^2 \bar{W}_R}{d\xi^2} \right] \\
& + \frac{dn}{d\xi} \frac{d\bar{W}_I}{d\xi} - m[(\zeta^2 - \omega^2 - \alpha^2 \sin^2 \theta) \bar{W}_I - 2\zeta\omega \bar{W}_R] = 0.
\end{aligned} \tag{32b}$$

At $\xi = 0$,

$$\begin{aligned}
& \gamma_{12} \bar{e} \frac{d^3 \bar{W}_R}{d\xi^3} + 2\gamma_{12} d_{qd} \left(\zeta \frac{d^2 \bar{W}_R}{d\xi^2} + \omega \frac{d^2 \bar{W}_I}{d\xi^2} \right) \\
& - \gamma_{12} a \left[(\zeta^2 - \omega^2 - \alpha^2) \frac{d\bar{W}_R}{d\xi} + 2\zeta\omega \frac{d\bar{W}_I}{d\xi} \right] + \gamma_{12} d_{hp} \frac{d\bar{W}_R}{d\xi} \\
& - \gamma_{12} d_{hd} \left[(\zeta^2 - \omega^2) \frac{d\bar{W}_R}{d\xi} + 2\zeta\omega \frac{d\bar{W}_I}{d\xi} \right] - \gamma_{12} n \frac{d\bar{W}_R}{d\xi} + \gamma_{11} \bar{W}_R = 0,
\end{aligned} \tag{33a}$$

$$\begin{aligned}
& \gamma_{12} \bar{e} \frac{d^3 \bar{W}_I}{d\xi^3} + 2\gamma_{12} d_{qd} \left(\zeta \frac{d^2 \bar{W}_I}{d\xi^2} - \omega \frac{d^2 \bar{W}_R}{d\xi^2} \right) \\
& - \gamma_{12} a \left[(\zeta^2 - \omega^2 - \alpha^2) \frac{d\bar{W}_I}{d\xi} - 2\zeta\omega \frac{d\bar{W}_R}{d\xi} \right] + \gamma_{12} d_{hp} \frac{d\bar{W}_I}{d\xi} \\
& - \gamma_{12} d_{hd} \left[(\zeta^2 - \omega^2) \frac{d\bar{W}_I}{d\xi} - 2\zeta\omega \frac{d\bar{W}_R}{d\xi} \right] - \gamma_{12} n \frac{d\bar{W}_I}{d\xi} + \gamma_{11} \bar{W}_I = 0,
\end{aligned} \tag{33b}$$

$$\gamma_{22} \bar{e} \frac{d^2 \bar{W}_R}{d\xi^2} + (\gamma_{22} d_{qd} \zeta - \gamma_{22} d_{qp} - \gamma_{21}) \frac{d\bar{W}_R}{d\xi} + \gamma_{22} d_{qd} \omega \frac{d\bar{W}_I}{d\xi} = 0, \tag{34a}$$

$$\gamma_{22} \bar{e} \frac{d^2 \bar{W}_I}{d\xi^2} + (\gamma_{22} d_{qd} \zeta - \gamma_{22} d_{qp} - \gamma_{21}) \frac{d\bar{W}_I}{d\xi} - \gamma_{22} d_{qd} \omega \frac{d\bar{W}_R}{d\xi} = 0. \tag{34b}$$

At $\xi = 1$,

$$\bar{e} \frac{d^2 \bar{W}_R}{d\xi^2} + (d_{qd} \zeta - d_{qp}) \frac{d\bar{W}_R}{d\xi} + d_{qd} \omega \frac{d\bar{W}_I}{d\xi} = 0, \tag{35a}$$

$$\bar{e} \frac{d^2 \bar{W}_I}{d\xi^2} + (d_{qd} \zeta - d_{qp}) \frac{d\bar{W}_I}{d\xi} - d_{qd} \omega \frac{d\bar{W}_R}{d\xi} = 0, \tag{35b}$$

$$\begin{aligned}
& \bar{e} \frac{d^3 \bar{W}_R}{d\xi^3} + 2d_{qd} \left(\zeta \frac{d^2 \bar{W}_R}{d\xi^2} + \omega \frac{d^2 \bar{W}_I}{d\xi^2} \right) - a \left[(\zeta^2 - \omega^2 - \alpha^2) \frac{d\bar{W}_R}{d\xi} + 2\zeta\omega \frac{d\bar{W}_I}{d\xi} \right] \\
& + d_{hp} \frac{d\bar{W}_R}{d\xi} - d_{hd} \left[(\zeta^2 - \omega^2) \frac{d\bar{W}_R}{d\xi} + 2\zeta\omega \frac{d\bar{W}_I}{d\xi} \right] = 0,
\end{aligned} \tag{36a}$$

$$\begin{aligned}
& \bar{e} \frac{d^3 \bar{W}_I}{d\xi^3} + 2d_{qd} \left(\zeta \frac{d^2 \bar{W}_I}{d\xi^2} - \omega \frac{d^2 \bar{W}_R}{d\xi^2} \right) - a \left[(\zeta^2 - \omega^2 - \alpha^2) \frac{d\bar{W}_I}{d\xi} - 2\zeta\omega \frac{d\bar{W}_R}{d\xi} \right] \\
& + d_{hp} \frac{d\bar{W}_I}{d\xi} - d_{hd} \left[(\zeta^2 - \omega^2) \frac{d\bar{W}_I}{d\xi} - 2\zeta\omega \frac{d\bar{W}_R}{d\xi} \right] = 0.
\end{aligned} \tag{36b}$$

3.2. Frequency equations

The fundamental solution of the characteristic differential equations (32) is assumed to be

$$\begin{bmatrix} \bar{W}_R(\xi) \\ \bar{W}_I(\xi) \end{bmatrix} = \sum_{i=1}^8 C_i \begin{bmatrix} \bar{W}_{R,i}(\xi) \\ \bar{W}_{I,i}(\xi) \end{bmatrix}, \tag{37}$$

where the eight linearly independent fundamental solutions $[\bar{W}_{R,i}(\xi) \ \bar{W}_{I,i}(\xi)]^T, i = 1,2,\dots,8$, of Eqs. (32) are chosen such that they satisfy the following normalization conditions at the origin of the coordinated system:

$$\begin{bmatrix} \bar{W}_{R,1} & \bar{W}_{R,2} & \bar{W}_{R,3} & \bar{W}_{R,4} & \bar{W}_{R,5} & \bar{W}_{R,6} & \bar{W}_{R,7} & \bar{W}_{R,8} \\ \bar{W}'_{R,1} & \bar{W}'_{R,2} & \bar{W}'_{R,3} & \bar{W}'_{R,4} & \bar{W}'_{R,5} & \bar{W}'_{R,6} & \bar{W}'_{R,7} & \bar{W}'_{R,8} \\ \bar{W}''_{R,1} & \bar{W}''_{R,2} & \bar{W}''_{R,3} & \bar{W}''_{R,4} & \bar{W}''_{R,5} & \bar{W}''_{R,6} & \bar{W}''_{R,7} & \bar{W}''_{R,8} \\ \bar{W}'''_{R,1} & \bar{W}'''_{R,2} & \bar{W}'''_{R,3} & \bar{W}'''_{R,4} & \bar{W}'''_{R,5} & \bar{W}'''_{R,6} & \bar{W}'''_{R,7} & \bar{W}'''_{R,8} \\ \bar{W}_{I,1} & \bar{W}_{I,2} & \bar{W}_{I,3} & \bar{W}_{I,4} & \bar{W}_{I,5} & \bar{W}_{I,6} & \bar{W}_{I,7} & \bar{W}_{I,8} \\ \bar{W}'_{I,1} & \bar{W}'_{I,2} & \bar{W}'_{I,3} & \bar{W}'_{I,4} & \bar{W}'_{I,5} & \bar{W}'_{I,6} & \bar{W}'_{I,7} & \bar{W}'_{I,8} \\ \bar{W}''_{I,1} & \bar{W}''_{I,2} & \bar{W}''_{I,3} & \bar{W}''_{I,4} & \bar{W}''_{I,5} & \bar{W}''_{I,6} & \bar{W}''_{I,7} & \bar{W}''_{I,8} \\ \bar{W}'''_{I,1} & \bar{W}'''_{I,2} & \bar{W}'''_{I,3} & \bar{W}'''_{I,4} & \bar{W}'''_{I,5} & \bar{W}'''_{I,6} & \bar{W}'''_{I,7} & \bar{W}'''_{I,8} \end{bmatrix}_{\xi=0} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \tag{38}$$

where primes indicate differentiation with respect to the dimensionless spatial variable ξ .

Substituting solution (37) into the boundary conditions (33)–(35) and the initial conditions (31), the following relation is obtained:

$$\begin{bmatrix} B_{11} & B_{12} & B_{13} & B_{14} & B_{15} & B_{16} & B_{17} & B_{18} \\ B_{21} & B_{22} & B_{23} & B_{24} & B_{25} & B_{26} & B_{27} & B_{28} \\ B_{31} & B_{32} & B_{33} & B_{34} & B_{35} & B_{36} & B_{37} & B_{38} \\ B_{41} & B_{42} & B_{43} & B_{44} & B_{45} & B_{46} & B_{47} & B_{48} \\ B_{51} & B_{52} & B_{53} & B_{54} & B_{55} & B_{56} & B_{57} & B_{58} \\ B_{61} & B_{62} & B_{63} & B_{64} & B_{65} & B_{66} & B_{67} & B_{68} \\ \bar{W}_{R,1}(1) & \bar{W}_{R,2}(1) & \bar{W}_{R,3}(1) & \bar{W}_{R,4}(1) & \bar{W}_{R,5}(1) & \bar{W}_{R,6}(1) & \bar{W}_{R,7}(1) & \bar{W}_{R,8}(1) \\ \bar{W}_{I,1}(1) & \bar{W}_{I,2}(1) & \bar{W}_{I,3}(1) & \bar{W}_{I,4}(1) & \bar{W}_{I,5}(1) & \bar{W}_{I,6}(1) & \bar{W}_{I,7}(1) & \bar{W}_{I,8}(1) \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \\ C_5 \\ C_6 \\ C_7 \\ C_8 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ w_0 \\ \frac{-1}{\omega}(\zeta w_0 + \dot{w}_0) \end{bmatrix}, \tag{39a}$$

where

$$\begin{aligned}
\mathbf{B}_{11} &= 0, \mathbf{B}_{12} = \gamma_{22}d_{qd}\zeta - \gamma_{22}d_{qp} - \gamma_{21}, \mathbf{B}_{13} = \gamma_{22}\bar{e}, \mathbf{B}_{14} = \mathbf{B}_{15} = 0, \\
\mathbf{B}_{16} &= \gamma_{22}d_{qd}\omega, \mathbf{B}_{17} = \mathbf{B}_{18} = 0, \\
\mathbf{B}_{21} &= 0, \mathbf{B}_{22} = -\gamma_{22}d_{qd}\omega, \mathbf{B}_{23} = \mathbf{B}_{24} = \mathbf{B}_{25} = 0, \mathbf{B}_{26} = \gamma_{22}d_{qd}\zeta - \gamma_{22}d_{qp} - \gamma_{21}, \mathbf{B}_{27} = \gamma_{22}\bar{e}, \mathbf{B}_{28} = 0; \\
\mathbf{B}_{31} &= \gamma_{11}, \\
\mathbf{B}_{32} &= -\gamma_{12}a(\zeta^2 - \omega^2 - \alpha^2) - \gamma_{12}d_{hd}(\zeta^2 - \omega^2) + \gamma_{12}d_{hp} - \gamma_{12}\alpha^2m(r + \frac{1}{2}), \\
\mathbf{B}_{33} &= 2\gamma_{12}d_{qd}\zeta, \mathbf{B}_{34} = \gamma_{12}\bar{e}, \mathbf{B}_{35} = 0, \\
\mathbf{B}_{36} &= -2\gamma_{12}a\zeta\omega - 2\gamma_{12}d_{hd}\zeta\omega, \\
\mathbf{B}_{37} &= 2\gamma_{12}d_{qd}\omega, \quad \mathbf{B}_{38} = 0, \\
\mathbf{B}_{41} &= 0, \quad \mathbf{B}_{42} = 2\gamma_{12}a\zeta\omega + 2\gamma_{12}d_{hd}\zeta\omega, \\
\mathbf{B}_{43} &= -2\gamma_{12}d_{qd}\omega, \quad \mathbf{B}_{44} = 0, \quad \mathbf{B}_{45} = \gamma_{11}, \\
\mathbf{B}_{46} &= -\gamma_{12}a(\zeta^2 - \omega^2 - \alpha^2) - \gamma_{12}d_{hd}(\zeta^2 - \omega^2) + \gamma_{12}d_{hp} - \gamma_{12}\alpha^2m(r + 1/2), \\
\mathbf{B}_{47} &= 2\gamma_{12}d_{qd}\zeta, \quad \mathbf{B}_{48} = \gamma_{12}\bar{e}, \\
B_{5i} &= \bar{e}\bar{W}''_{R,i}(1) + (d_{qd}\zeta - d_{qp})\bar{W}'_{R,i}(1) + d_{qd}\omega\bar{W}'_{I,i}(1), \\
B_{6i} &= \bar{e}\bar{W}''_{I,i}(1) + (d_{qd}\zeta - d_{qp})\bar{W}'_{I,i}(1) - d_{qd}\omega\bar{W}'_{R,i}(1), \quad i = 1, 2, \dots, 8.
\end{aligned} \tag{39b}$$

Given the initial displacement w_0 and velocity \dot{w}_0 , the oscillating frequency and the decay rate can be easily determined via Eqs. (36), called as a coupled frequency equations, by using the numerical method proposed by Lin [14].

3.3. Exact fundamental solutions

In general, the closed-form fundamental solutions of two coupled differential equations with variable coefficients are not available. However, if the coefficients of the equations, which involve the material properties and/or geometric parameters, can be expressed in matrix polynomial form, then a power series representation of the fundamental solutions can be constructed by the modified method of Frobenius [6]. Eqs. (32) can be expressed as

$$\begin{aligned}
\bar{A}_1 \frac{d^4 \bar{W}_R}{d\xi^4} + \bar{A}_2 \frac{d^3 \bar{W}_R}{d\xi^3} + \bar{A}_3 \frac{d^2 \bar{W}_R}{d\xi^2} + \bar{A}_4 \frac{d \bar{W}_R}{d\xi} + \bar{A}_5 \bar{W}_R \\
+ \bar{A}_6 \frac{d^3 \bar{W}_I}{d\xi^3} + \bar{A}_7 \frac{d^2 \bar{W}_I}{d\xi^2} + \bar{A}_8 \bar{W}_I = 0,
\end{aligned} \tag{40a}$$

$$\begin{aligned}
\tilde{A}_1 \frac{d^4 \bar{W}_I}{d\xi^4} + \tilde{A}_2 \frac{d^3 \bar{W}_I}{d\xi^3} + \tilde{A}_3 \frac{d^2 \bar{W}_I}{d\xi^2} + \tilde{A}_4 \frac{d \bar{W}_I}{d\xi} + \tilde{A}_5 \bar{W}_I \\
+ \tilde{A}_6 \frac{d^3 \bar{W}_R}{d\xi^3} + \tilde{A}_7 \frac{d^2 \bar{W}_R}{d\xi^2} + \tilde{A}_8 \bar{W}_R = 0, \quad \xi \in (0, 1),
\end{aligned} \tag{40b}$$

where

$$\begin{aligned}
\bar{A}_1 &= a_0, & \tilde{A}_1 &= \bar{a}_0, \\
\bar{A}_2 &= b_0, & \tilde{A}_2 &= \bar{b}_0, \\
\bar{A}_3 &= c_0 + c_1\xi + c_2\xi^2, & \tilde{A}_3 &= \bar{c}_0 + \bar{c}_1\xi + \bar{c}_2\xi^2, \\
\bar{A}_4 &= d_0 + d_1\xi, & \tilde{A}_4 &= \bar{d}_0 + \bar{d}_1\xi, \\
\bar{A}_5 &= e_0, & \tilde{A}_5 &= \bar{e}_0,
\end{aligned}$$

$$\begin{aligned}\bar{A}_6 &= f_0, \quad \tilde{A}_6 = \bar{f}_0, \\ \bar{A}_7 &= g_0, \quad \tilde{A}_7 = \bar{g}_0, \\ \bar{A}_8 &= h_0, \quad \tilde{A}_8 = \bar{h}_0\end{aligned}$$

in which

$$\begin{aligned}a_0 &= \bar{a}_0 = -\bar{e}, \quad b_0 = \bar{b}_0 = -2d_{qd}\zeta, \\ c_0 &= \bar{c}_0 = a(\zeta^2 - \omega^2 - \alpha^2) - d_{hp} + d_{hd}(\zeta^2 - \omega^2) + \alpha^2 m(r + 1/2), \\ c_1 &= \bar{c}_1 = -r\alpha^2 m, \quad c_2 = \bar{c}_2 = -\frac{1}{2}\alpha^2 m, \\ d_0 &= \bar{d}_0 = -r\alpha^2 m, \quad d_1 = \bar{d}_1 = -\alpha^2 m, \\ e_0 &= \bar{e}_0 = -m(\zeta^2 - \omega^2 - \alpha^2 \sin^2 \theta), \quad f_0 = -\bar{f}_0 = -2d_{qd}\omega, \\ g_0 &= -\bar{g}_0 = 2\zeta\omega a + 2\zeta\omega d_{hd}, \quad h_0 = -\bar{h}_0 = -2\zeta\omega m.\end{aligned}\quad (41b)$$

The eight normalized fundamental solutions of Eqs. (32) are expressed as

$$\begin{bmatrix} \bar{W}_{R,j} \\ \bar{W}_{I,j} \end{bmatrix} = \sum_{k=0}^{\infty} \begin{bmatrix} \alpha_{j,k} \zeta^k \\ \beta_{j,k} \zeta^k \end{bmatrix}, \quad j = 1, 2, \dots, 8, \quad (42)$$

which can be derived by using the recurrence formula given by Lin et al. [7]. Consequently, upon substituting these fundamental solutions into the frequency equation (36), the exact complex frequencies of the beam are obtained.

4. Numerical results and discussion

To demonstrate the efficiency and convergence of the proposed method to solve the vibration problem, the transient response, the frequency shift and the decay rate of a rotating beam are determined. Fig. 2 demonstrates the transient response of the beam. The initial displacement and velocity are given. The oscillation of the beam decays away exponentially. In Table 1, the convergence pattern of the complex eigenvalues of a rotating beam is shown. It shows that the eigenvalues determined by the proposed method converge very rapidly. The convergent frequencies without the piezoelectric damping are the same as those given by Lin [14].

Fig. 3 shows influence of the gain factors k_d and k_p on the frequencies ω_i and the decay rates ζ_i of the first two modes of a rotating beam with an elastic root. Figs. 3a and 3c show that without the proportional and derivative controls, i.e., $k_d = k_p = 0$, the frequencies of the first two modes are the natural frequencies of free vibration of a rotating beam. In the neglect of the derivative control, i.e., $k_d = 0$, decreasing the proportional gain factor increases the frequencies of oscillation. Given any proportional gain factor, whatever the derivative gain factors is increased or decreased from zero, the frequencies of oscillation are decreased. Figs. 3b and 3d show that in the neglect of the derivative control, i.e., $k_d = 0$, whatever the proportional gain factor is given, the decay rates $\{\zeta_1, \zeta_2\}$ are zero. In other words, if $k_d = 0$, considering the proportional control only is not helpful to the active damping of a rotating beam.

Fig. 3b shows that if $k_p = 0$, considering a positive or negative derivative gain factor will results in a positive or negative decay rate ζ_1 of the first mode, respectively. It means that if the decay rate is positive, the amplitude decay exponentially and the system is stable. Otherwise, it is unstable. Increasing the derivative gain factor k_d from zero, the decay rate ζ_1 increases rapidly from zero to a critical value and then decreases slowly. This fact happens also to the first mode of a cantilever beam [7]. If $k_p < 0$, increasing the negative proportional gain factor enhances the phenomenon. However, if $k_p > 0$, increasing the positive proportional gain factor upsets the phenomenon. It is observed from Fig. 3b that in the neglect of the proportional control, if $k_d < 0$, the first decay rate is negative. In other words, the oscillation of the first mode is divergent. But it is observed from

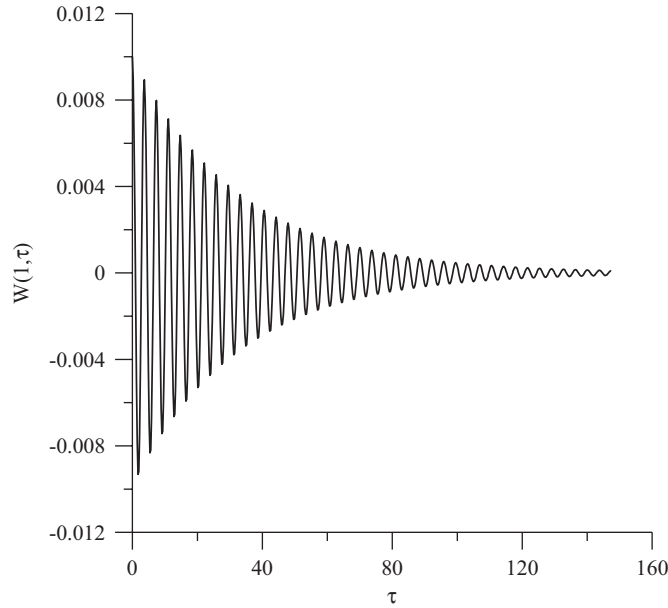


Fig. 2. Oscillation of a rotating beam [$k_d = 0.3$, $k_p = 0$, $\gamma_{11} = 0.9$, $\gamma_{21} = 0.9$, $w_0 = 0.01$, $\dot{w}_0 = 0$, $a = 0.00001$, $e = 1.722$, $m = 1.807$, $d_h = 0.98k_d^2$, $d_q = 1.892k_d$, $r = 0.5$, $\alpha = 1$, $\theta = 30^\circ$].

Table 1

Convergence pattern of the first two eigenvalues of a rotating beam with an elastic root [$\gamma_{11} = 0.9$, $\gamma_{21} = 0.9$, $w_0 = 0.001$, $\dot{w}_0 = 0$, $a = 0.00001$, $e = m = 1$, $dh = 0.98k_d^2$, $d_q = 1.892k_d$, $r = 0.5$, $\alpha = 1$, $\theta = 30^\circ$]

No. of terms	$k_d = k_p = 0$		$k_d = 0.1, k_p = -0.05$			
	ω_1	ω_2	ω_1	ζ_1	ω_2	ζ_2
15	2.2487	6.9254	2.2321	0.0269	5.7386	0.1147
20	2.2487	6.9254	2.2321	0.0269	5.7386	0.1149
30	2.2487	6.9254	2.2321	0.0269	5.7386	0.1149
40	2.2487	6.9254	2.2321	0.0269	5.7386	0.1149
50	2.2487	6.9254	2.2321	0.0269	5.7386	0.1149
[14]	2.2487	6.9258	–	–	–	–

Fig. 3d that if $k_d < -0.3$, the second decay rate becomes positive. It means that the oscillation of the first mode is convergent. Moreover, It is observed from Fig. 3b that if $k_d > 0$, the first decay rate is positive. In other words, the oscillation of the first mode will be convergent. But it is observed from Fig. 3d that if $k_d > 0.3$, the second decay rate become negative. It means that the oscillation of the second mode will be divergent. It is concluded that considering the active damping of the first two modes, the derivative gain factor k_d should be limited.

Fig. 4 shows the influences of the gain factors k_d and k_p and the rotational spring constant γ_{21} on the frequency and the decay rate of the first mode. It is observed from Fig. 4b that if $k_d = 0$, whatever the rotational spring constant and the proportional gain factor are, the decay rate is zero. If $k_d = 1$, increasing the rotational spring constant increases the first decay rate. Moreover, It is observed from Fig. 4a that increasing the rotational spring constant increases the first frequency of oscillation especially for the case of $k_d = 0$.

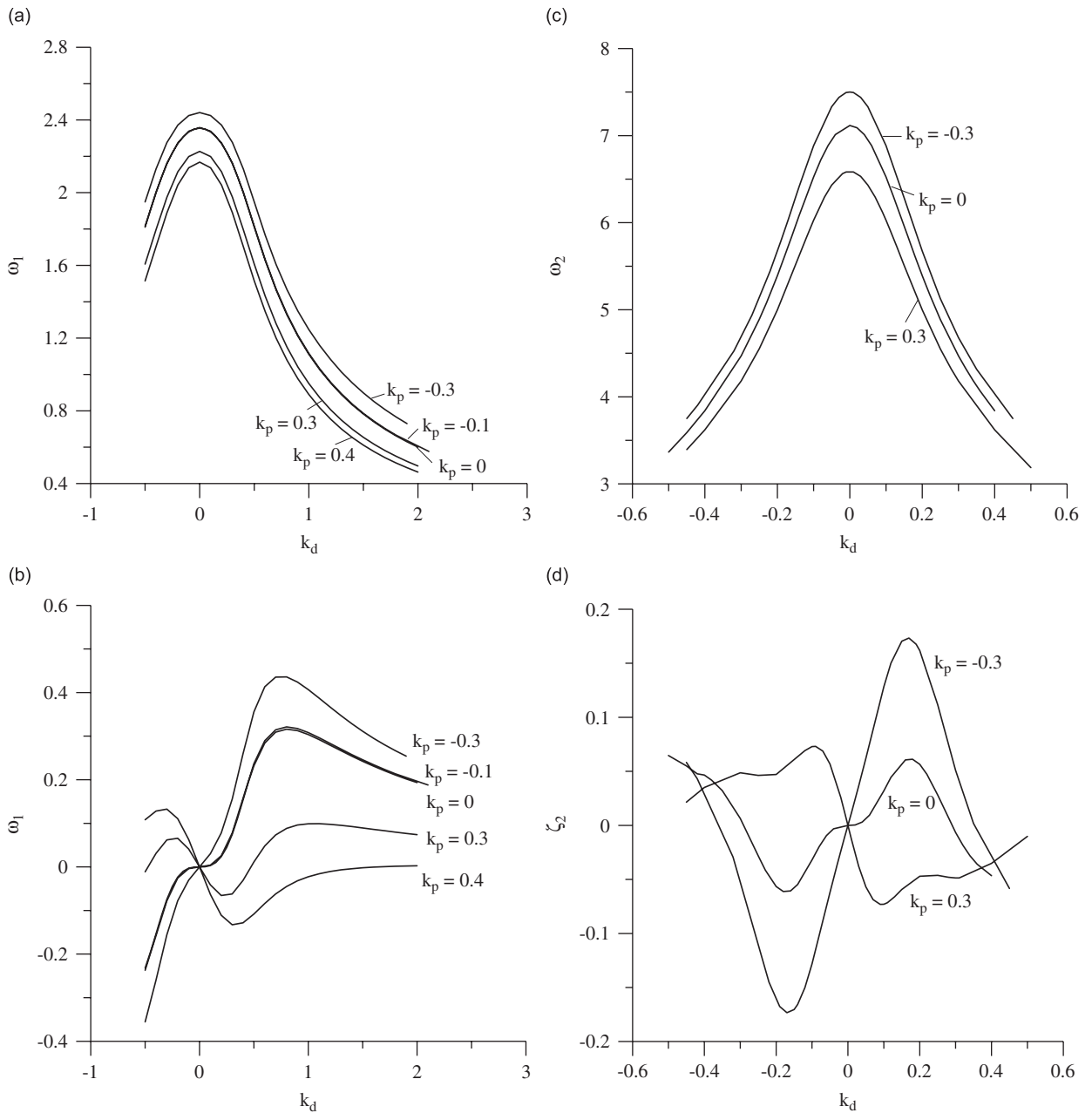


Fig. 3. Influence of the gain factors k_d and k_p on the frequencies ω_i and the decay rates ζ_i of the first two modes of a rotating beam [$\gamma_{11} = 0.95, \gamma_{21} = 0.95, \omega_0 = 0.01, \dot{\omega}_0 = 0, a = 0.00001, e = 1.722, m = 1.807, d_h = 0.98k_d^2, d_q = 1.892k_d, r = 0.5, \theta = 30^\circ$].

It is well known that a rigid root of a rotating beam is considered, i.e., $\gamma_{11} = \gamma_{21} = 1$, increasing the rotating speed increases the frequency. However, if an elastic root is considered and the root spring constants are small enough, increasing the rotating speed decreases the first frequency [14]. When the rotating speed increases to a critical value, the first frequency decreases to zero. Figs. 5a and 5b show the influences of the gain factors k_d and k_p and the rotating speed α on the frequency and the decay rate of the first mode. It is observed from Fig. 5a that the influence of the proportional gain factor on the first frequency is small. But as shown in Fig. 5b, Its effect on the first decay rate is large. Moreover, increasing the rotating speed α greatly decreases the

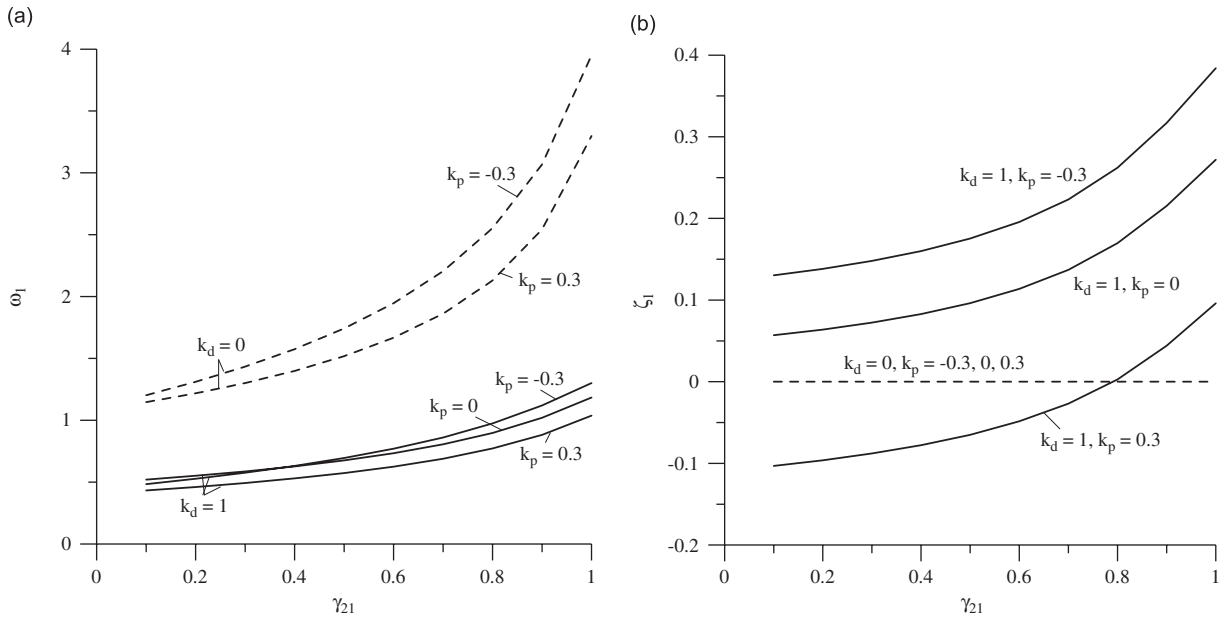


Fig. 4. Influence of the gain factors k_d and k_p and the rotational spring constant γ_{21} on the frequencies ω_1 and the decay rate ζ_1 of the first mode of a rotating beam [$\gamma_{11} = 1$, $w_0 = 0.01$, $\dot{w}_0 = 0$, $a = 0.00001$, $e = 1.722$, $m = 1.807$, $d_n = 0.98k_d^2$, $d_q = 1.892k_d$, $r = 0.5$, $\theta = 30^\circ$].

first decay rate of a rotating beam with an elastic root. It is different to that of a cantilever beam with a clamped root. Lin et al. [7] found that increasing the rotating speed increases the first decay rate of a rotating beam with a clamped root.

Fig. 5c shows the influence of the proportional gain factor k_p and the rotating speed α on the Q -factor. The definition of the Quality factor is $Q\text{-factor} = 2\pi E_t / E_{\text{loss}}$ where E_t is the total energy and E_{loss} is the loss energy per cycle [7]. When the rotating speed α is very small, the Q -factor is almost constant and small. It means that the energy dissipation is very large. As shown in Fig. 5b, if $k_p = 0$ and the rotating speed approaches a critical value, the decay rate approaches zero. In other words, the energy dissipation is negligible and the Q -factor approaches infinite, as shown in Fig. 5c. The Q -factor of a rotating beam with an elastic root obviously depends on the proportional gain factor and the rotating speed.

5. Conclusion

In this paper, the proportional and derivative control laws are simultaneously applied to the active damping of the first two oscillating modes of a rotating beam. It has been found by Lin et al. [7] that the derivative control law is helpful to the active damping of the first oscillating mode. But considering the proportional control law only is unuseful for the active damping. However, considering the proportional and derivative controls suitably and simultaneously will enhance the active damping of a rotating beam with an elastic root. Moreover, it is also found that:

- (1) Considering the active damping of any mode, the derivative gain factor should be limited.
- (2) Increasing the rotating speed α greatly decreases the first decay rate of a rotating beam with an elastic root. But increasing the rotating speed increases the first decay rate of a rotating beam with a clamped root.
- (3) The Q -factor of a rotating beam with an elastic root obviously depends on the proportional gain factor and the rotating speed.

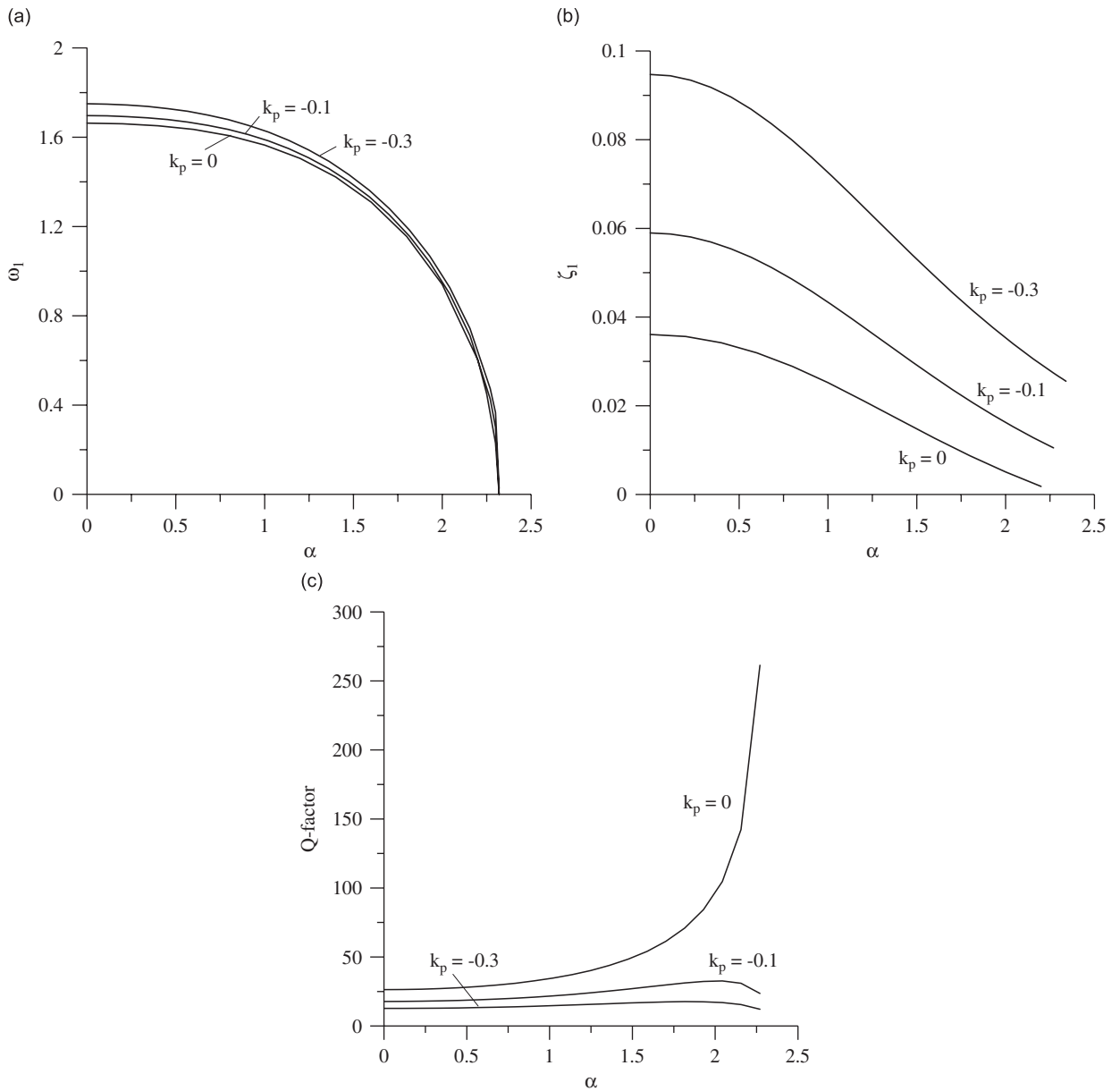


Fig. 5. Influence of the rotational speed α and the proportional gain k_p on the frequency ω_1 , the Q -factor and the decay rate ζ_1 of a rotating beam [$k_d = 0.3$, $\gamma_{11} = 0.9$, $\gamma_{21} = 0.9$, $w_0 = 0.01$, $\dot{w}_0 = 0$, $a = 0.00001$, $e = 1.722$, $m = 1.807$, $d_h = 0.98k_d^2$, $d_q = 1.892k_d$, $r = 0.5$, $\theta = 60^\circ$].

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References

- [1] A. Leissa, Vibrational aspects of rotating turbomachinery blades, *Applied Mechanics Reviews—Transactions of the ASME* 34 (5) (1981) 629–635.
- [2] V. Ramamurti, P. Balasubramanian, Analysis of turbomachine blades: a review, *The Shock and Vibration Digest* 16 (16) (1984) 13–28.

- [3] J.S. Rao, Turbomachine blade vibration, *The Shock and Vibration Digest* 19 (5) (1987) 3–10.
- [4] S.M. Lin, The instability and vibration of rotating beams with arbitrary pretwist and an elastically restrained root, *Journal of Applied Mechanics—Transactions of the ASME* 68 (6) (2001) 844–853.
- [5] S.M. Lin, S.Y. Lee, W.R. Wang, Dynamic analysis of rotating damped beams with an elastically restrained root, *International Journal of Mechanical Science* 46 (2004) 673–693.
- [6] S.M. Lin, S.Y. Lee, Prediction of vibration and instability of rotating damped beams with an elastically restrained root, *International Journal of Mechanical Sciences* 46 (2004) 1173–1194.
- [7] S.M. Lin, I.C. Mao, J.H. Lin, Vibration of a rotating smart beam, *AIAA Journal* 45 (2) (2007) 382–389.
- [8] J.S. Turcotte, J.J. Hollkamp, R.W. Gordon, Vibration of a mistuned bladed-disk assembly using structurally damped beams, *AIAA Journal* 36 (12) (1998) 2225–2228.
- [9] B.P. Patel, M. Ganapathi, Non-linear torsional vibration and damping analysis of sandwich beams, *Journal of Sound and Vibration* 240 (2) (2001) 385–393.
- [10] M.I. Friswell, A.W. Lees, The mode of non-homogeneous damped beams, *Journal of Sound and Vibration* 242 (2) (2001) 355–361.
- [11] U. Lee, J. Kim, Spectral element modeling for beams treated with active constrained layer damping, *International Journal of Solids and Structures* 38 (2001) 5679–5702.
- [12] A. Benjeddou, Advances in piezoelectric finite element modeling of adaptive structural elements: a survey, *Computers and Structures* 76 (2000) 347–363.
- [13] D. Sun, L. Tong, Modeling and analysis of curved beams with debonded piezoelectric sensor/actuator patches, *International Journal of Mechanical Sciences* 44 (2002) 1755–1777.
- [14] S.M. Lin, Dynamic analysis of rotating nonuniform Timoshenko beam with an elastically restrained root, *Journal of Applied Mechanics—Transactions of the ASME* 66 (3) (1999) 742–749.